Solving Life-Cycle Models with a Rich Asset Structure using Deep Learning

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based on joint work with Jan Žemlička,[2,3] Luca Gaegauf,[2] and Simon Scheidegger[4,5]

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Motivation

- Economic models to study questions related to aggregate risk and asset pricing, often require global solution methods to compute equilibria
- Computing a functional rational expectations equilibrium amounts to computing a set of functions, f_i, mapping the state of the economy, $\mathbf{x}$, to endogenous outcomes $f_i(\mathbf{x})$:

$$f_i : \mathcal{D} \subset \mathbb{R}^{N_{\text{in}}} \to \mathbb{R} : \underbrace{\mathbf{x}}_{\text{state}} \to \underbrace{f_i(\mathbf{x})}_{\text{endogenous variables}}, \text{ s.t. : } \underbrace{\mathbf{G}(\mathbf{x}, f_1, \ldots, f_{N_{\text{out}}}) = 0}_{\text{equilibrium conditions}}$$

- This can be a computationally demanding task, especially when
  - the state of the economy is high-dimensional
  - the equilibrium functions are nonlinear
- Both often happens for Overlapping Generations (OLG) models:
  - the state includes the wealth distribution across age-groups
  - young households are often constrained
  - may want to account for portfolio decomposition and volatility of labor income, both of which have strong lifecycle components

This talk

- Basic solution method developed in Azinovic et al. (2022)
- More recent progress on portfolio choice and market clearing neural network architectures developed in Azinovic and Žemlička (2023)

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- Other papers on deep learning based solution methods I learned a lot form: Maliar et al. (2021); Kase et al. (2023); Gu et al. (2023); Kahou et al. (2021); Han et al. (2022); Valaitis and Villa (2024); Kahou et al. (2022); Fernández-Villaverde et al. (2023); Barnett et al. (2023); Jungerman (2023); Kahou et al. (2024)

Deep Equilibrium Nets

Violations of equilibrium conditions as loss function

Basic idea in Azinovic et al. (2022): write equilibrium conditions as

$$\mathbf{G}(\mathbf{x}, \mathbf{f}) = 0 \ \forall \mathbf{x}$$

$$\begin{aligned} \mathbf{G} &: \text{ equilibrium conditions: FOC's, market clearing, Bellman equations, } \ldots \\ \mathbf{x} &: \text{ state of the economy} \\ \mathbf{f} &: \text{ equilibrium functions.} \end{aligned}$$

Approximate $\mathbf{f}$ by neural network $\mathcal{N}_\rho$

$$\mathcal{N}_\rho(\mathbf{x}) \approx \mathbf{f}(\mathbf{x})$$

How?

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$$\mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x})$$

How?

Standard deep learning:

- need labeled data, i.e. inputs for which we know the true output: $\{\mathbf{x}_i, f(\mathbf{x}_i)\}_i$
- train neural network parameters $\boldsymbol{\rho}$ to minimize the loss function

$$\ell_{\boldsymbol{\rho}} := \frac{1}{N_{\text{labeled data}}} \sum_{\mathbf{x}_i} \left(f(\mathbf{x}_i) - \mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}_i)\right)^2$$

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Deep equilibrium nets:

- use equilibrium conditions directly as loss function

$$\ell_{\boldsymbol{\rho}} := \frac{1}{N_{\text{path length}}} \sum_{\mathbf{x}_i \text{ on sim. path}} \left(\mathbf{G}(\mathbf{x}_i, \mathcal{N}_{\boldsymbol{\rho}})\right)^2$$

- no need for labeled data! What are Neural Nets? Why use Neural Nets?

Training DEQNs

1. Simulate a sequence of states $\mathcal{D}^i_{\text{train}} \leftarrow \{\mathbf{x}^i_1, \mathbf{x}^i_2, \ldots, \mathbf{x}^i_T\}$ from the policy encoded by the network parameters $\boldsymbol{\rho}^i$.
2. Evaluate the errors of the equilibrium conditions on the newly generated set $\mathcal{D}_{\text{train}}$.
3. If the error statistics are not low enough:
   3.1 update the parameters of the neural network with a gradient descent step (or a variant):
   $$\rho^{i+1}_k = \rho^i_k - \alpha_{\text{learn}} \frac{\partial \ell_{\mathcal{D}^i_{\text{train}}}(\boldsymbol{\rho}^i)}{\partial \rho^i_k}.$$
   3.2 set new starting states for simulation: $\mathbf{x}^{i+1}_0 = \mathbf{x}^i_T$.
   3.3 increase i by one and go back to step 1.

Illustrative Model

Illustrative OLG model with capital and bond

- Representative firm produces with

$$
\begin{aligned}
F(z_t, K_t, L) &= z_t K_t^{\alpha} L^{1-\alpha} \\
w_t &= \alpha z_t K_t^{\alpha-1} L^{1-\alpha} \\
r_t &= z_t (1-\alpha) K_t^{\alpha} L^{\alpha}
\end{aligned}
$$

- Uncertainty in TFP z_t, and depreciation of capital δ_t

$$
\begin{aligned}
\log(z_{t+1}) &= \rho_z \log(z_t) + \sigma_z \epsilon_t \\
\epsilon_t &\sim N(0,1) \\
\delta_t &= \delta \frac{2}{1+z}
\end{aligned}
$$

- Assets
  - one period bond with price p_t in aggregate supply B
  - risky capital K_t
  - borrowing constraints on both assets

$$
\begin{aligned}
b_t^h &\geq 0 \\
k_t^h &\geq 0
\end{aligned}
$$

- Households
  - $H = 32$ age-groups, indexed with $h \in \mathcal{H} := \{1, \ldots, 32\}$
  - supply labor units l_t^h inelastically
  - adjustment costs on capital

$$
\begin{aligned}
\Delta_{k,t}^h &:= k_{t+1}^{h+1} - k_t^h \\
\text{adj. costs} &= \psi \left(\Delta_{k,t}^h \right)^2
\end{aligned}
$$

  - budget constraint

$$
\begin{aligned}
c_t^h = {} & l^h w_t + b_{t-1}^{h-1} + k_{t-1}^{h-1}(1 - \delta_t + r_t) \\
& - p_t^b b_t^h - k_t^h - \psi \left(\Delta_{k,t}^h \right)^2
\end{aligned}
$$

  - maximize

$$
\mathrm{E}\left[\sum_{i=h}^{H} \beta^{i-h} u(c_{t+i}^{h+i}) \right]
$$

$$
u(c) := \frac{c^{1-\gamma} - 1}{1-\gamma}
$$

Equilibrium conditions

- **Market clearing**:

$$K_t := \sum_{h \in \mathcal{H}} k_t^h$$

$$B = \sum_{h \in \mathcal{H}} b_t^h \Leftrightarrow \epsilon_t^B := B - \sum_{h \in \mathcal{H}} b_t^h = 0$$

- **Firms optimize**:

$$w_t := \alpha z_t K_t^{\alpha - 1} L^{1-\alpha}$$

$$r_t := z_t (1-\alpha) K_t^{\alpha} L^{\alpha}$$

- **Households optimize**:
  - H sets of Karush Kuhn Tucker conditions for bond
    $\Rightarrow$ single equation using the Fisher-Burmeister equation
    $\Rightarrow$ H errors $\epsilon_t^{k,i}$
  - H sets of Karush Kuhn Tucker conditions for capital
    $\Rightarrow$ single equation using the Fisher-Burmeister equation
    $\Rightarrow$ H errors $\epsilon_t^{h,i}$

details

Approximation with standard DEQN

- State of the economy

$$\mathbf{x}_t = [\underbrace{z_t}_{\text{ex. shock}}, \underbrace{k_t^1, \dots, k_t^{32}}_{\text{dist. of cap.}}, \underbrace{b_t^1, \dots, b_t^{32}}_{\text{dist. of bonds}}]$$

- Equilibrium policies

$$\mathbf{f}(\mathbf{x}_t) = [\underbrace{k_{t+1}^1, \dots, k_{t+1}^{32}}_{\text{capital policy}}, \underbrace{b_{t+1}^1, \dots, b_{t+1}^{32}}_{\text{bond policy}}, \underbrace{p_t^b}_{\text{bond price}}]$$

- Neural network approximates

$$\mathcal{N}_\rho(\mathbf{x}_t) = [\underbrace{\hat{k}_{t+1}^1, \dots, \hat{k}_{t+1}^{32}}_{\text{capital policy}}, \underbrace{\hat{b}_{t+1}^1, \dots, \hat{b}_{t+1}^{32}}_{\text{bond policy}}, \underbrace{\hat{p}_t^b}_{\text{bond price}}] \approx \mathbf{f}(\mathbf{x}_t)$$

- Loss function

$$\ell_\rho(\mathbf{x}_t) := \underbrace{w_{hh,k}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon_t^{k,h}\right)^2\right)}_{\text{opt. cond. cap.}} + \underbrace{w_{hh,b}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon_t^{b,h}\right)^2\right)}_{\text{opt. cond. bond}} + \underbrace{w_{mc,B}}_{\text{weight}} \underbrace{\left(\epsilon_t^{B}\right)^2}_{\text{market clearing}}$$

Innovation 1: Market clearing layers

- Neural network first predicts

$$\mathcal{N}^{\text{pre}}_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, \tilde{b}^1_{t+1}, \ldots, \tilde{b}^{32}_{t+1}, \hat{p}^b_t]$$

- Apply transformation $m(\ldots, \cdot)$

$$[\hat{b}^1_{t+1}, \ldots, \hat{b}^{32}_{t+1}] = m\left(\mathcal{N}^{\text{pre}}_{\boldsymbol{\rho}}(\mathbf{x}_t), B\right)$$

- Such that

$$B = \sum_{h=1}^{32} \hat{b}^h_{t+1}$$

- Put together

$$\mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}_t) := [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, \hat{b}^1_{t+1}, \ldots, \hat{b}^{32}_{t+1}, \hat{p}^b_t]$$

- Loss function now

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1. no need to learn economics we already know ex-ante
2. remaining loss easier to interpret
3. states simulated from the policy are always consistent with market clearing ▸ details

Innovation 2: Stabilizing step-wise model transformations

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- Many asset models are hard
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  - portfolio choice only pinned down at low errors in equilibrium conditions
  - but how do we get there?

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$$\mathcal{N}_{\rho}^{1}(\mathbf{x}_t) = [\hat{k}_{t+1}^{1}, \ldots, \hat{k}_{t+1}^{32}, 0\times\hat{b}_{t+1}^{1}, \ldots, 0\times\hat{b}_{t+1}^{32}, \hat{p}_t^{b}], B^{1} = 0$$

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  3. solve the model

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  3. solve the model
  4. train the neural network to predict the bond price (supervised, from zero liquidity limit)
  5. slowly introduce the second asset (such that the error remains low)

$$\mathcal{N}_{\boldsymbol{\rho}}^{2}(\mathbf{x}_t) = [\hat{k}_{t+1}^{1}, \ldots, \hat{k}_{t+1}^{32}, 1\times\hat{b}_{t+1}^{1}, \ldots, 1\times\hat{b}_{t+1}^{32}, \hat{p}_t^{b}], B^2 = 0.1$$

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$$\mathcal{N}_{\boldsymbol{\rho}}^{3}(\mathbf{x}_t) = [\hat{k}_{t+1}^{1}, \ldots, \hat{k}_{t+1}^{32}, 1 \times \hat{b}_{t+1}^{1}, \ldots, 1 \times \hat{b}_{t+1}^{32}, \hat{p}_t^b], B^3 = 0.2$$

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$$\mathcal{N}^1_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 0\times\hat{b}^1_{t+1}, \ldots, 0\times\hat{b}^{32}_{t+1}, \hat{p}^b_t], B^1 = 0$$

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$$\mathcal{N}^4_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 1\times\hat{b}^1_{t+1}, \ldots, 1\times\hat{b}^{32}_{t+1}, \hat{p}^b_t], B^4 = 0.3$$

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$$\mathcal{N}^5_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 1 \times \hat{b}^1_{t+1}, \ldots, 1 \times \hat{b}^{32}_{t+1}, \hat{p}^b_t], B^5 = 0.4$$

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$$\mathcal{N}^{\cdots}_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 1 \times \hat{b}^1_{t+1}, \ldots, 1 \times \hat{b}^{32}_{t+1}, \hat{p}^b_t], B^{\cdots} = \ldots$$

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  3. solve the model
  4. train the neural network to predict the bond price (supervised, from zero liquidity limit)
  5. slowly introduce the second asset (such that the error remains low)

$$\mathcal{N}_{\boldsymbol{\rho}}^{100}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 1 \times \hat{b}^1_{t+1}, \ldots, 1 \times \hat{b}^{32}_{t+1}, \hat{p}^b_t], B^{100} = 10$$

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  6. equilibrium errors always remain low

Application

Step 1: Solve single asset model

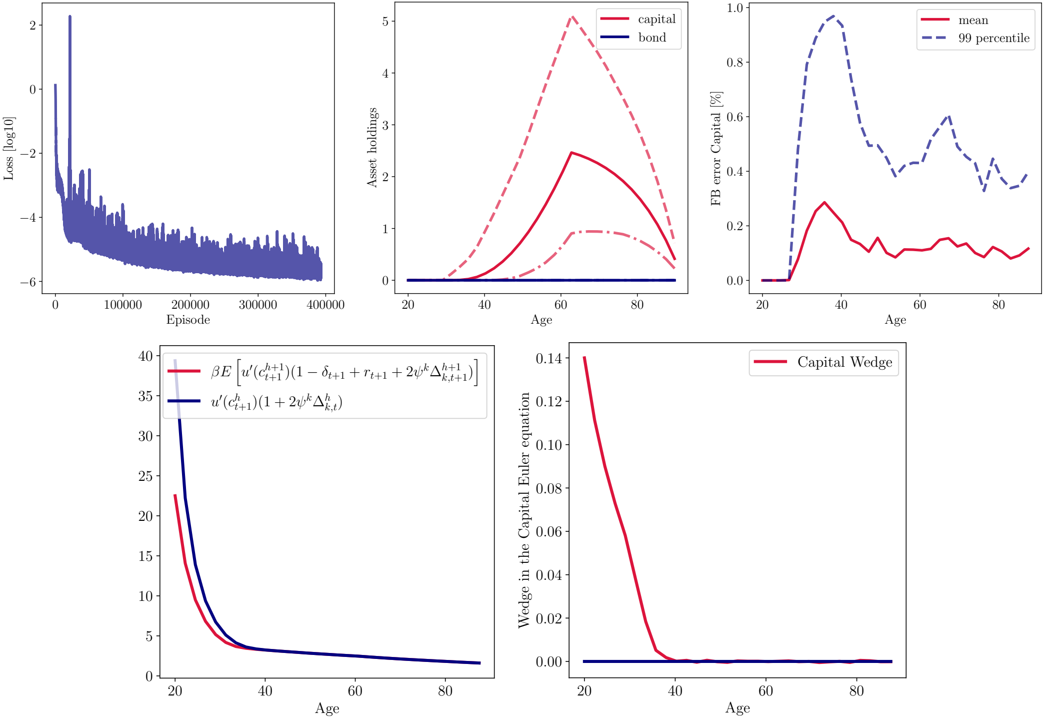
- Borrowing constraint $\underline{b} = 0$, net-supply $B = 0$
- Neural network predicts

$$\mathcal{N}_{\boldsymbol{\rho}}^{\text{pre}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 0 \times \tilde{b}^1_{t+1}, \ldots, 0 \times \tilde{b}^{32}_{t+1}, \hat{p}^b_t]$$

$$\Rightarrow \mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 0, \ldots, 0, \hat{p}^b_t]$$

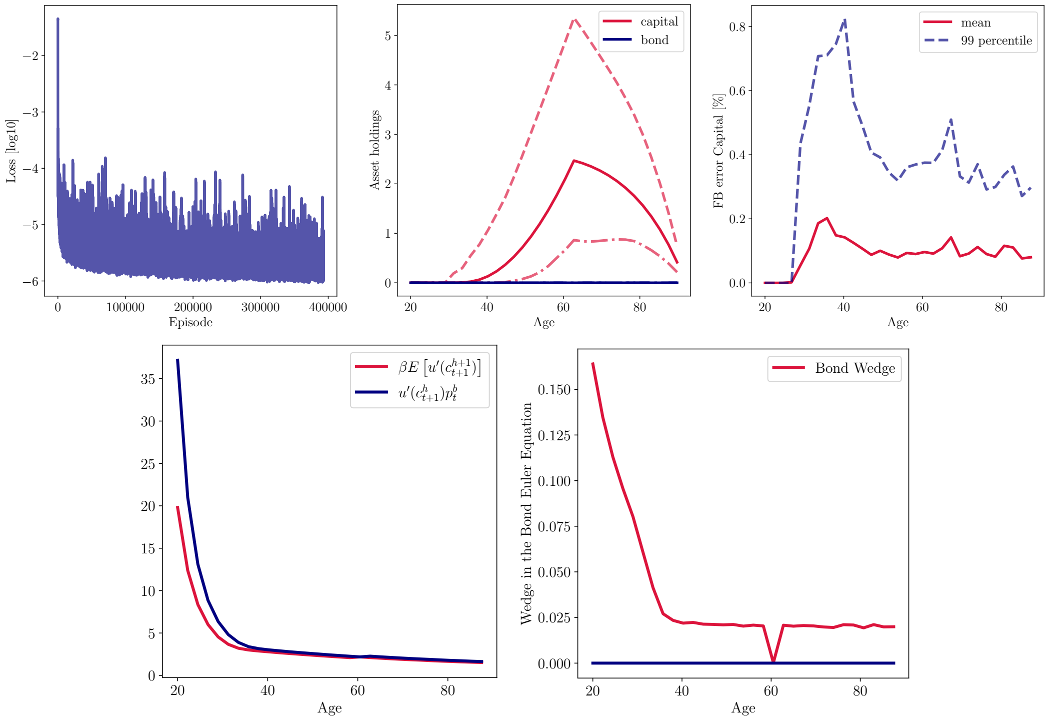
- Loss function

$$\ell_{\boldsymbol{\rho}}(\mathbf{x}_t) := 1 \times \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon_t^{k,h} \right)^2 \right)}_{\text{opt. cond. cap.}} + \underbrace{0 \times \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon_t^{b,h} \right)^2 \right)}_{\text{opt. cond. bond}}}_{=0}$$


Loss [log10]
Episode
Asset holdings
capital
bond
Age
FB error Capital [%]
mean
99 percentile
$\beta E\left[u'(c_{t+1}^{h+1})(1-\delta_{t+1}+r_{t+1}+2\psi^k\Delta_{k,t+1}^{h+1})\right]$
$u'(c_{t+1}^{h})(1+2\psi^k\Delta_{k,t}^{h})$
Wedge in the Capital Euler equation
Capital Wedge

Step 2: Pre-train bond price in the capital only model

- Keep borrowing constraint $\underline{b} = 0$, net-supply $B = 0$, and neural network masks
$$\mathcal{N}_\rho(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 0, \ldots, 0, \hat{p}^b_t]$$
- In equilibrium we know that
$$p^b_t \geq \frac{\beta \mathrm{E}\left[u'(c^{h+1}_{t+1})\right]}{u'(c^h_t)}$$
with equality for unconstrained agents.
- With market clearing policies, we have a closed form expression for the bond price and can define pre-train price and error
$$\begin{aligned} p^{b,\text{pre-train}}_t &:= \max_{h \in \mathcal{H}} \left\{ \frac{\beta \mathrm{E}\left[u'(c^{h+1}_{t+1})\right]}{u'(c^h_t)} \right\} \\ \epsilon^{\text{pre-train}}_t &:= p^{b,\text{pre-train}}_t - \hat{p}^b_t \end{aligned}$$
- Loss function
$$\ell_\rho(\mathbf{x}_t) := 1 \times \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon^{k,h}_t\right)^2 \right)}_{\text{opt. cond. cap.}} + 0 \times \underbrace{\underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon^{b,h}_t\right)^2 \right)}_{\text{opt. cond. bond}}}_{=0} + 1 \times \underbrace{\left(\epsilon^{\text{pre-train}}_t\right)^2}_{\substack{\text{price pre-train error} \\ \text{train supervised}}}$$


Loss [log10]
Episode
Asset holdings
capital
bond
Age
FB error Capital [%]
mean
99 percentile
Age
$\beta E\left[u'(c_{t+1}^{h+1})\right]$
$u'(c_{t+1}^{h})p_t^b$
Age
Wedge in the Bond Euler Equation
Bond Wedge
Age

Step 3: Slowly increase bond supply

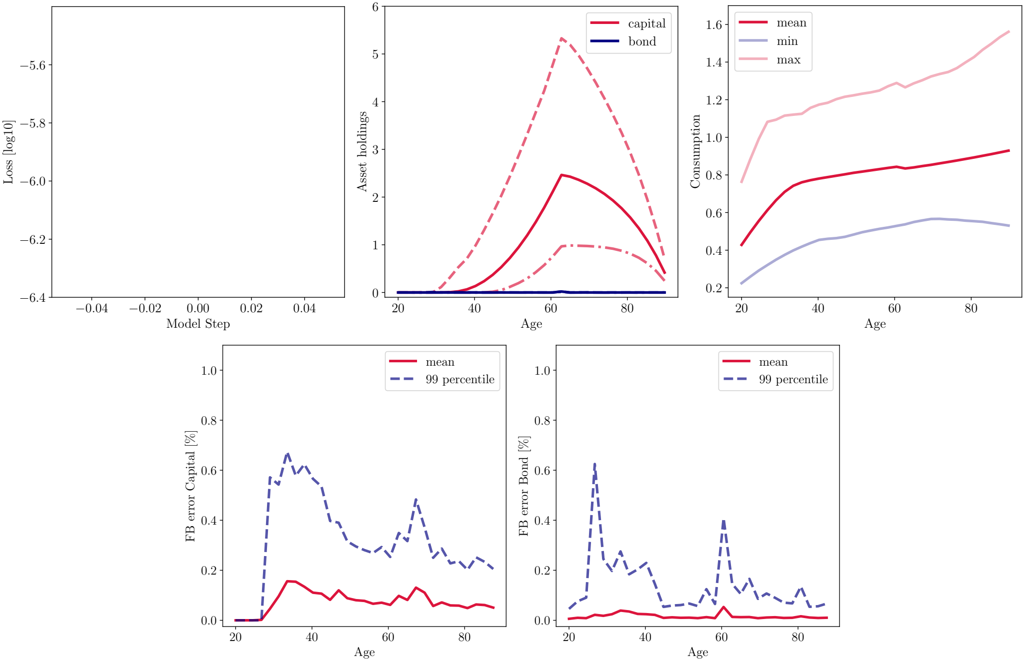
- Borrowing constraint $\underline{b} = 0$, increase net-supply from $B = 0.1$ to $B = 10$
- Neural network predicts

$$\mathcal{N}_{\boldsymbol{\rho}}^{\text{pre}}(\mathbf{x}_t) = [\hat{k}_{t+1}^{1}, \ldots, \hat{k}_{t+1}^{32}, \underbrace{0.01 \times \tilde{b}_{t+1}^{1}, \ldots, 0.01 \times \tilde{b}_{t+1}^{32}}_{\text{bond policies active}}, \hat{p}_t^b]$$

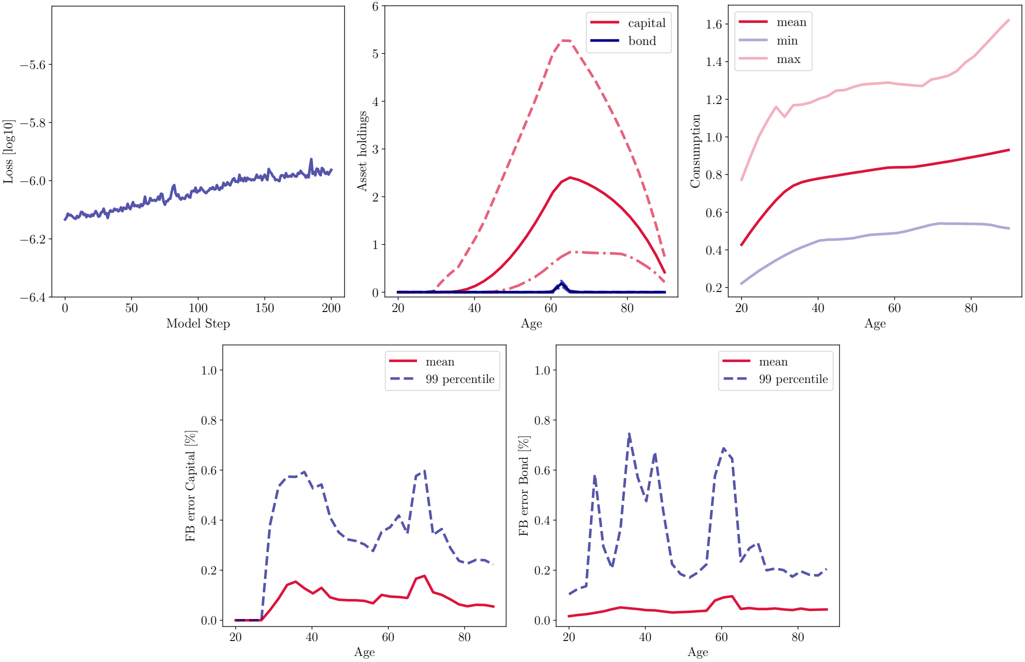
$$\Rightarrow \mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}_{t+1}^{1}, \ldots, \hat{k}_{t+1}^{32}, \underbrace{\hat{b}_{t+1}^{1}, \ldots, \hat{b}_{t+1}^{32}}_{\text{always add up the B}}, \hat{p}_t^b]$$

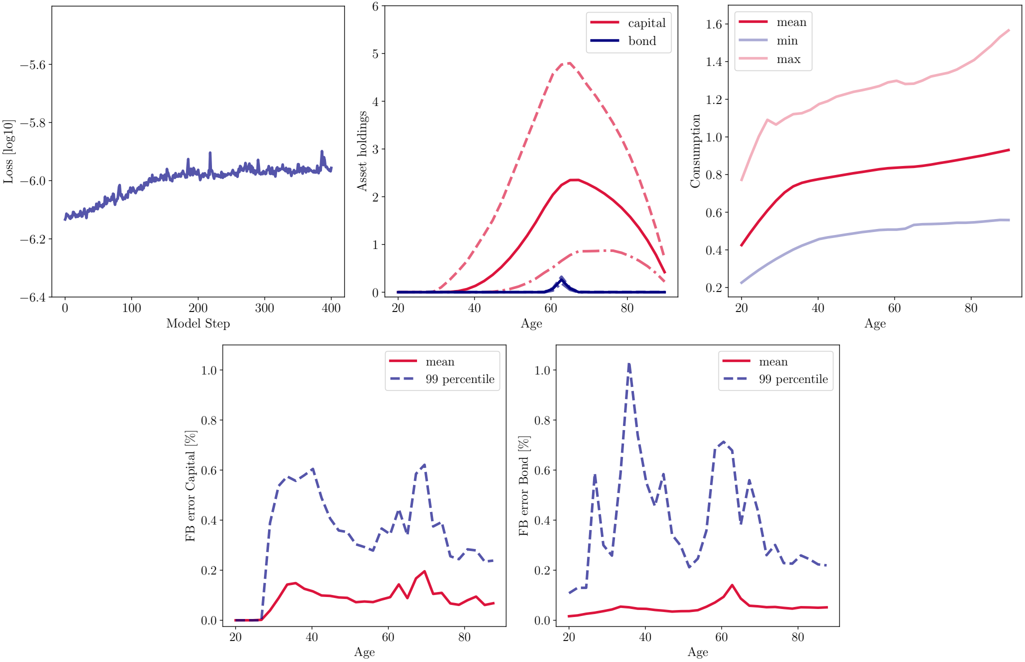
- Loss function

$$\ell_{\boldsymbol{\rho}}(\mathbf{x}_t) := 1 \times \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon_t^{k,h} \right)^2 \right)}_{\text{opt. cond. cap.}} + \underbrace{1 \times}_{\text{bond equ. cond. active}} \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon_t^{b,h} \right)^2 \right)}_{\text{opt. cond. bond}}$$

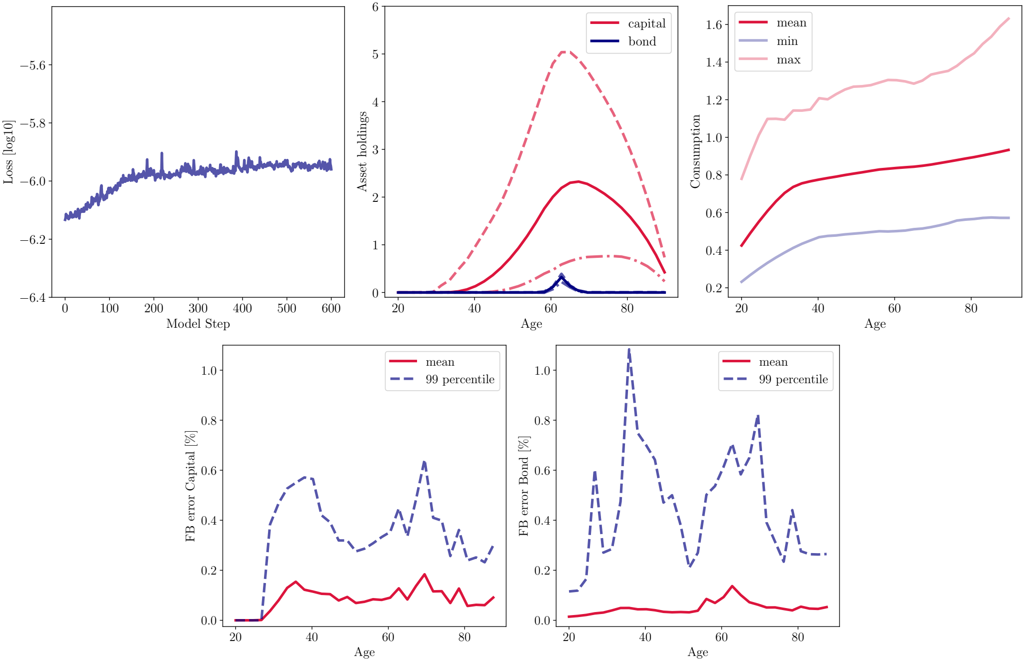


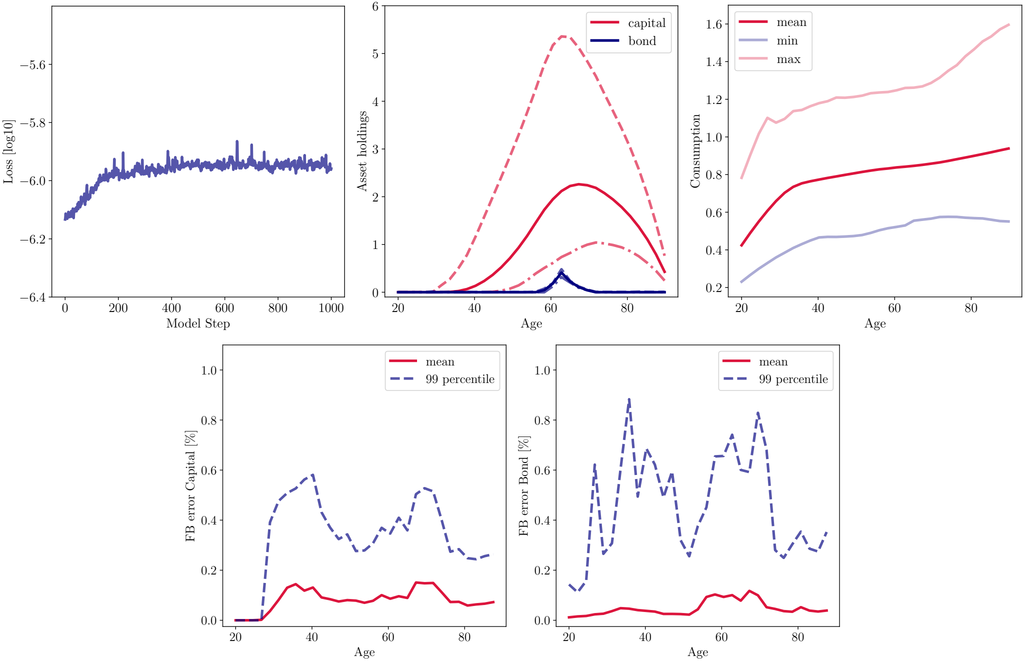
Loss [log10]
Model Step
Asset holdings
Age
capital
bond
Consumption
mean
min
max
FB error Capital [%]
99 percentile
FB error Bond [%]

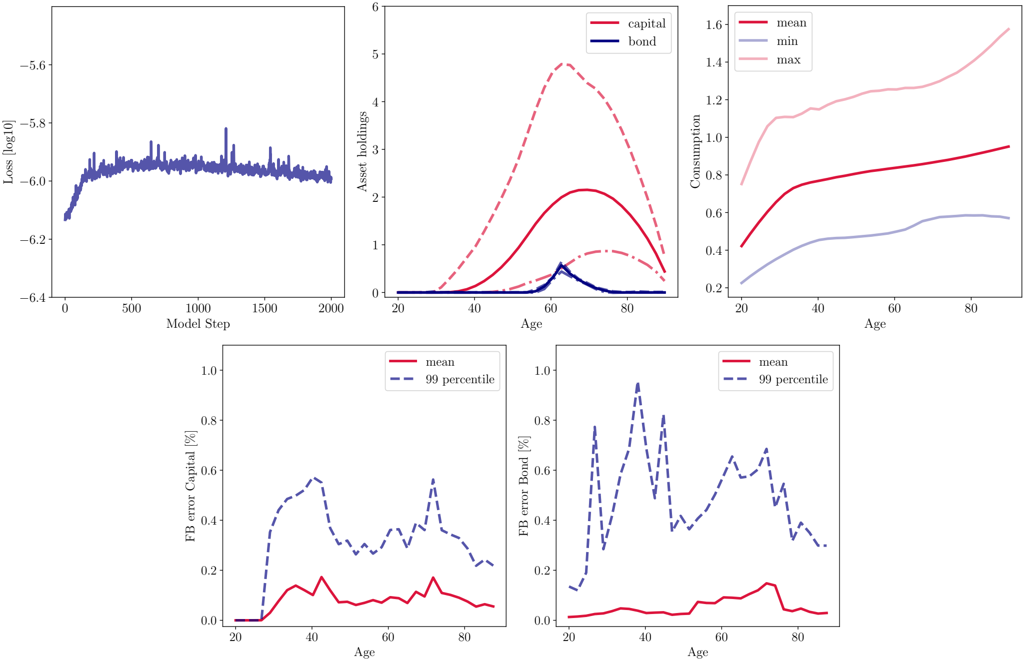

Loss [log10]
Model Step
capital
bond
Asset holdings
Age
mean
min
max
Consumption
Age
mean
99 percentile
FB error Capital [%]
Age
mean
99 percentile
FB error Bond [%]
Age



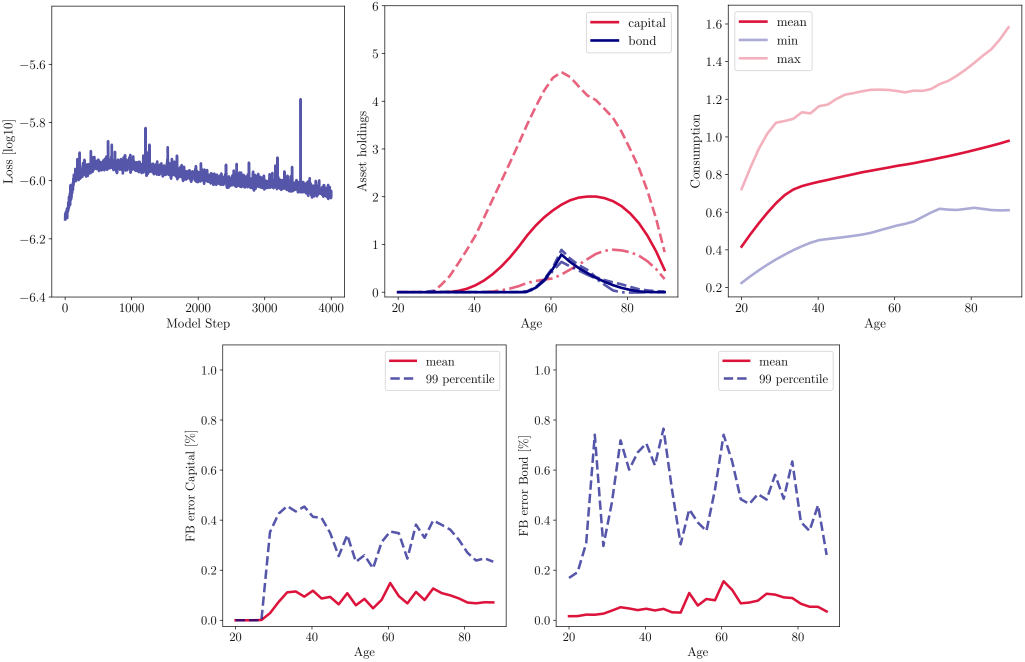
Loss [log10]
Model Step
Asset holdings
Age
capital
bond
Consumption
mean
min
max
FB error Capital [%]
mean
99 percentile
FB error Bond [%]
mean
99 percentile

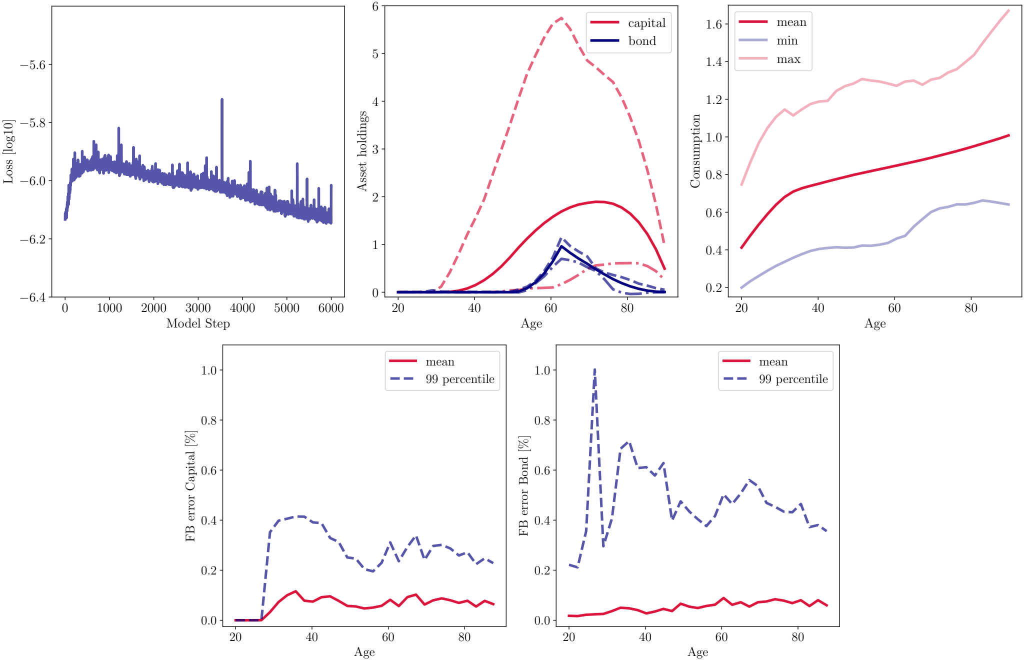

Loss [log10]
Model Step
Asset holdings
Age
capital
bond
Consumption
mean
min
max
FB error Capital [%]
mean
99 percentile
FB error Bond [%]
mean
99 percentile

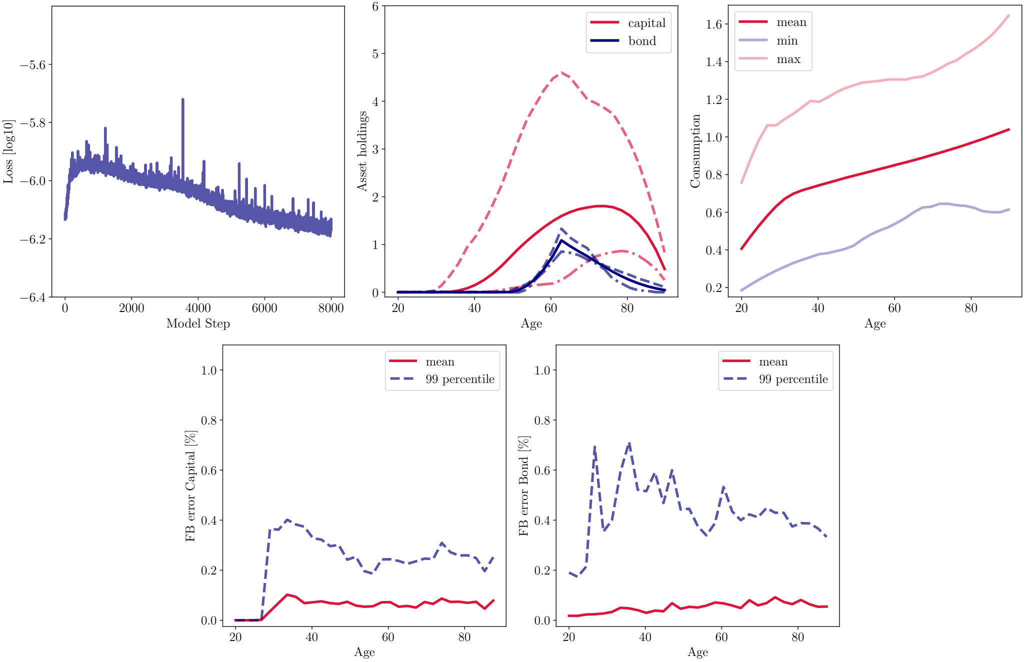

Loss [log10]
Model Step
−5.6
−5.8
−6.0
−6.2
−6.4
0
200
400
600
800
1000
Asset holdings
Age
capital
bond
Consumption
mean
min
max
FB error Capital [%]
mean
99 percentile
FB error Bond [%]
mean
99 percentile

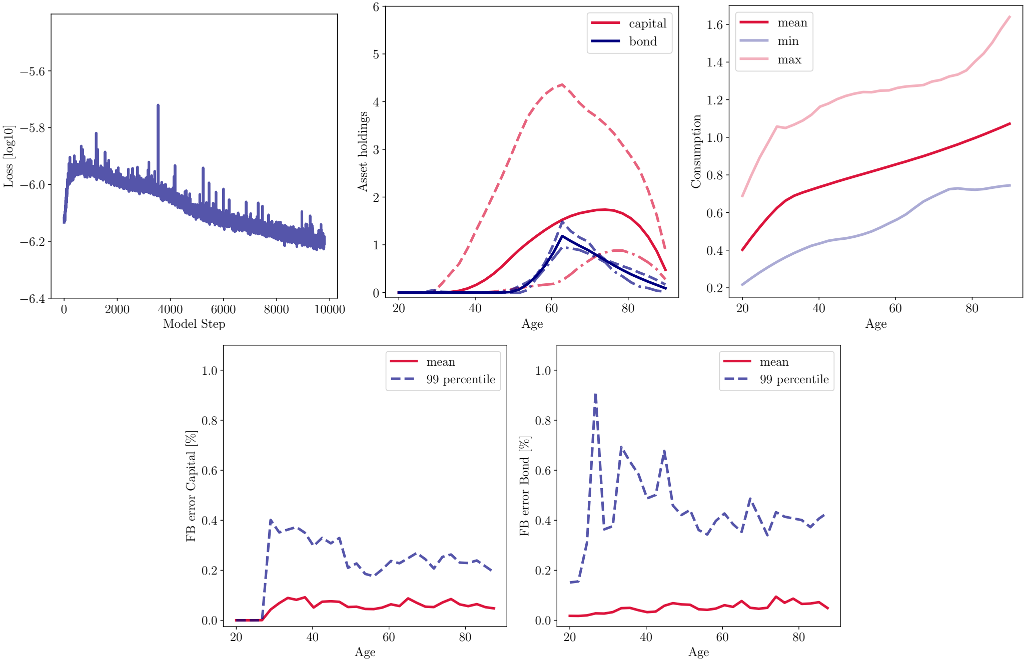


Loss [log10]
Model Step
capital
bond
Asset holdings
Age
mean
min
max
Consumption
Age
mean
99 percentile
FB error Capital [%]
Age
mean
99 percentile
FB error Bond [%]
Age


Loss [log10]
Model Step
Asset holdings
Age
capital
bond
Consumption
mean
min
max
FB error Capital [%]
mean
99 percentile
FB error Bond [%]


Loss [log10]
Model Step
capital
bond
Asset holdings
Age
mean
min
max
Consumption
Age
mean
99 percentile
FB error Capital [%]
Age
mean
99 percentile
FB error Bond [%]
Age


Loss [log10]
−5.6
−5.8
−6.0
−6.2
−6.4
0
2000
4000
6000
8000
Model Step
Asset holdings
capital
bond
Age
Consumption
mean
min
max
FB error Capital [%]
mean
99 percentile
FB error Bond [%]


Loss [log10]
Model Step
Asset holdings
capital
bond
Age
Consumption
mean
min
max
FB error Capital [%]
mean
99 percentile
FB error Bond [%]

Step 4: Training with the final supply

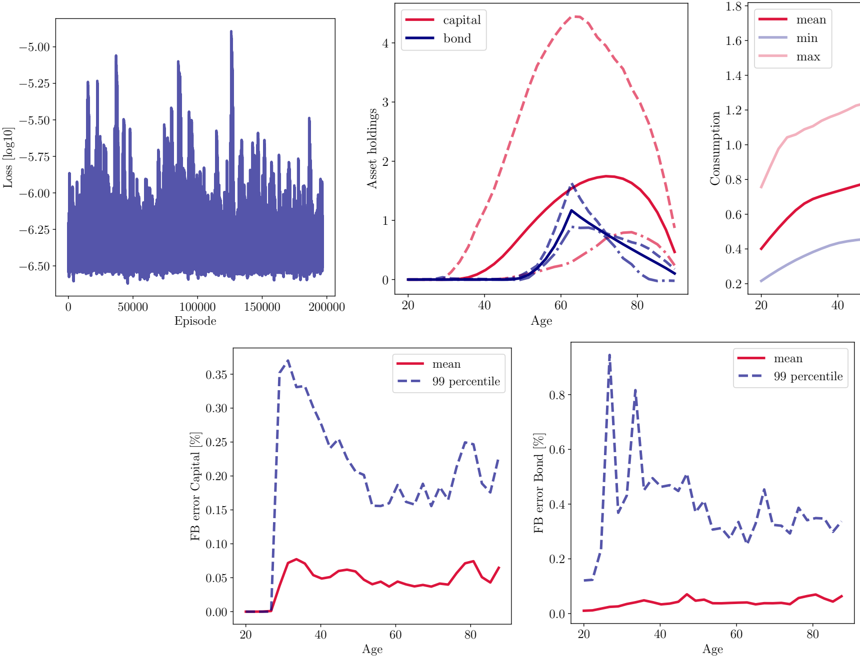
- Borrowing constraint $\underline{b} = 0$, bond at full net-supply from $B = 10$
- Neural network predicts

$$\mathcal{N}_{\rho}^{\text{pre}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, \underbrace{0.01\times\tilde{b}^1_{t+1}, \ldots, 0.01\times\tilde{b}^{32}_{t+1}}_{\text{bond policies active}}, \hat{p}^b_t]$$

$$\Rightarrow \mathcal{N}_{\rho}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, \underbrace{\hat{b}^1_{t+1}, \ldots, \hat{b}^{32}_{t+1}}_{\text{always add up the B}}, \hat{p}^b_t]$$

- Loss function contains all remaining equilibrium conditions

$$\ell_{\rho}(\mathbf{x}_t) := 1\times \underbrace{\left(\sum_{h=1}^{H-1}\left(\epsilon_t^{k,h}\right)^2\right)}_{\text{opt. cond. cap.}} + 1\times \underbrace{\left(\sum_{h=1}^{H-1}\left(\epsilon_t^{b,h}\right)^2\right)}_{\text{opt. cond. bond}}$$



Loss [log10]
−5.00
−5.25
−5.50
−5.75
−6.00
−6.25
−6.50
0
50000
100000
150000
200000
Episode
Asset holdings
capital
bond
0
1
2
3
4
20
40
60
80
Age
Consumption
mean
min
max
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
20
40
60
80
Age
FB error Capital [%]
mean
99 percentile
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
20
40
60
80
Age
FB error Bond [%]
mean
99 percentile
0.0
0.2
0.4
0.6
0.8
20
40
60
80
Age

Conclusion

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- Deep neural networks are promising to approximate nonlinear functions on high-dimensional domains
- Key ideas in Azinovic et al. (2022):
  - minimizing the error in the equilibrium conditions allows training the neural network without labeled data $\Rightarrow$ neural network can be trained on billions of states
  - training on the simulated path $\Rightarrow$ focus training on where it matters
- Models with many assets remain challenging. To address this issue Azinovic and Žemlička (2023) introduce key innovations
  - market clearing layers, an economics-inspired neutral network architecture
  - step-wise model transformation procedure to guide network training with many assets

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- Also in the paper: quantitative life-cycle model with disaster risk, housing, equity and bonds in general equilibrium to study the intergenerational consequences of rare disasters (updated version coming soon)
- Other cool papers on deep learning based solution methods: Maliar et al. (2021); Kase et al. (2023); Gu et al. (2023); Kahou et al. (2021); Han et al. (2022); Valaitis and Villa (2024); Kahou et al. (2022); Fernández-Villaverde et al. (2023); Barnett et al. (2023); Jungerman (2023); Kahou et al. (2024)

Thank you!

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Deep Neural Networks

What is a deep neural net?

Consider:

$$\textbf{input} := \mathbf{x} \to W_{\boldsymbol{\rho}}^{1}\mathbf{x} + \mathbf{b}_{\boldsymbol{\rho}}^{1} =: \textbf{hidden 1}$$

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Consider:

$$\begin{aligned} &\textbf{input} := \mathbf{x} \to W^1_{\boldsymbol{\rho}}\mathbf{x} + \mathbf{b}^1_{\boldsymbol{\rho}} =: \textbf{hidden 1} \\ &\to \textbf{hidden 1} \to W^2_{\boldsymbol{\rho}}(\textbf{hidden 1}) + \mathbf{b}^2_{\boldsymbol{\rho}} =: \textbf{hidden 2} \end{aligned}$$

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The parameters $\boldsymbol{\rho}$ of this procedure are the entries of the matrices $(W_{\boldsymbol{\rho}}^1,\ W_{\boldsymbol{\rho}}^2,\ W_{\boldsymbol{\rho}}^3)$ and vectors $(\mathbf{b}_{\boldsymbol{\rho}}^1,\ \mathbf{b}_{\boldsymbol{\rho}}^2,\ \mathbf{b}_{\boldsymbol{\rho}}^3)$.

What is a deep neural net? (cont.)

So far we have a concatenation of affine maps and therefore an afffine map.

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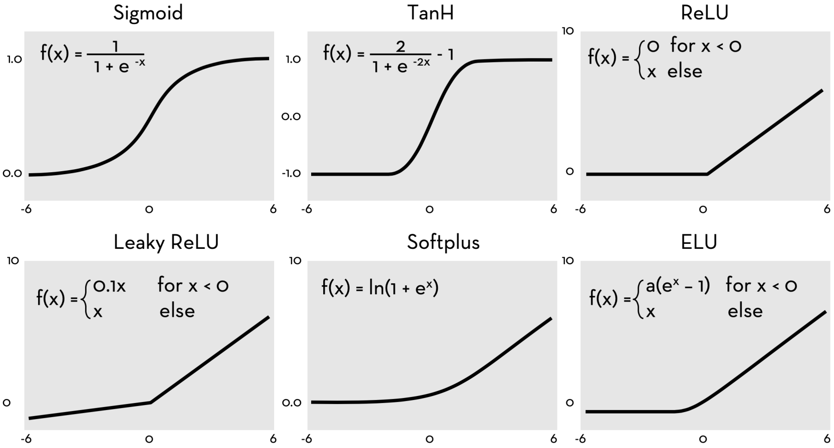
So far we have a concatenation of affine maps and therefore an afffine map.
Next ingredient: activation functions ϕ^1, ϕ^2, ϕ^3. Activation functions could be any function, but popular are:



What is a deep neural net? (cont.)

Now we get:

$$\begin{aligned} &\textbf{input} := \mathbf{x} \to \phi^1(W_{\boldsymbol{\rho}}^1\mathbf{x} + \mathbf{b}_{\boldsymbol{\rho}}^1) =: \textbf{hidden 1} \\ \to\ &\textbf{hidden 1} \to \phi^2(W_{\boldsymbol{\rho}}^2(\textbf{hidden 1}) + \mathbf{b}_{\boldsymbol{\rho}}^2) =: \textbf{hidden 2} \\ \to\ &\textbf{hidden 2} \to \phi^3(W_{\boldsymbol{\rho}}^3(\textbf{hidden 2}) + \mathbf{b}_{\boldsymbol{\rho}}^3) =: \textbf{output} \end{aligned}$$

The neural net is then given by the choice of activation functions and the parameters $\boldsymbol{\rho}$.

back

Why neural networks?

Approximation method	High-dimensional input	Can resolve local features accurately	Irregularly shaped domain	Large amount of data
Polynomials	✓	✗	✓	✓
Splines	✗	✓	✗	✓
Adaptive (sparse) grids	✓	✓	✗	✓
Gaussian processes	✓	✓	✓	✗
Deep neural networks	✓	✓	✓	✓

Table: Taken from Azinovic et al. (2022).

back

Innovation 1: Details on the market clearing transformation function

- Simple market clearing layer: subtract excess demand ED_t from initial predictions

$$ED_t := \sum_{h \in \mathcal{H}} \tilde{b}^h_{t+1} - B$$

$$\hat{b}^h_{t+1} := \tilde{b}^h_{t+1} - \frac{1}{H} ED_t$$

- Why this adjustment?

$\rightarrow$ we try to minimize the modification to the initial predictions $\{\tilde{b}^h_{t+1}\}_{h \in \mathcal{H}}$.

- Final predictions $\{\hat{b}^h_{t+1}\}_{h \in \mathcal{H}}$ solve

$$\underset{\{x^h_{t+1}\}_{h \in \mathcal{H}}}{\arg\min} \sum_{h \in \mathcal{H}} \left(x^h_{t+1} - \tilde{b}^h_{t+1} \right)^2$$

subject to

$$\sum_{h \in \mathcal{H}} x^h_{t+1} = B$$

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subject to

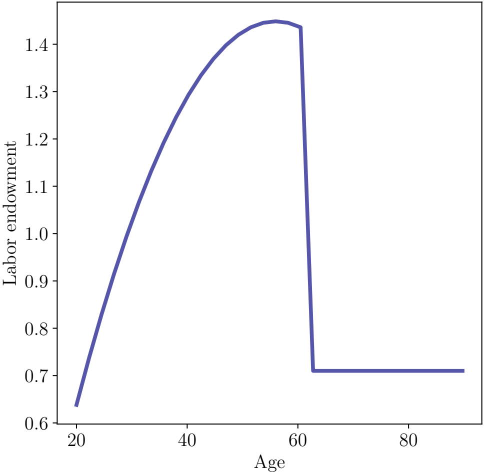
$$\sum_{h \in \mathcal{H}} x^h_{t+1} = B$$

- In the paper: enforcing market clearing & borrowing constraints using implicit layer

back

Parameters

Parameters	H	β	γ	ψ	ρ	σ	α
Values	32	0.912	4	0.1	0.693	0.052	0.333
Meaning	num. age groups	patience	RRA	adj. costs	pers. tfp	std. innov. tfp	cap. share



Households' optimality conditions

$$\left.\begin{aligned} 1 &= \frac{\beta \mathrm{E}\left[u'(c^{h+1}_{t+1})(1-\delta_{t+1}+r_{t+1}+2\psi^k\Delta^{h+1}_{k,t+1}\right]+\mu^h_t}{(1+2\psi^k\Delta^h_{k,t})u'(c^h_t)} \\ k^h_t &\geq 0 \\ \mu^h_t &\geq 0 \\ k^h_t\mu^h_t &= 0 \end{aligned}\right\} \Leftrightarrow \epsilon^{k,h}_t := \psi^{FB}\left(\frac{u'^{-1}\left(\beta \mathrm{E}\left[u'(c^{h+1}_{t+1})\frac{(1-\delta_{t+1}+r_{t+1}+2\psi^k\Delta^{h+1}_{k,t+1}}{(1+2\psi^k\Delta^h_{k,t})}\right]\right)}{c^h_t}-1, \frac{k^h_t}{c^h_t}\right)$$

$$\left.\begin{aligned} 1 &= \frac{\beta \mathrm{E}\left[u'(c^{h+1}_{t+1})\right]+\lambda^h_t}{p^b_t u'(c^h_t)} \\ b^h_t - \underline{b} &\geq 0 \\ \lambda^h_t &\geq 0 \\ (b^h_t - \underline{b}^h)\lambda^h_t &= 0 \end{aligned}\right\} \Leftrightarrow \epsilon^{b,h}_t := \psi^{FB}\left(\frac{u'^{-1}\left(\beta \mathrm{E}\left[\frac{1}{p^b_t}u'(c^{h+1}_{t+1})\right]\right)}{c^h_t}-1, \frac{b^h_t-\underline{b}}{c^h_t}\right)$$

where

$$\psi^{FB}(a, b) := a + b - \sqrt{a^2 + b^2}$$

back